Gaussian R.V. PPF

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left\{-\frac{(x-\bar{X})^2}{2\sigma_X^2}\right\}$$

$$\Phi_X(w) = e^{jw\bar{X} - \frac{1}{2}w^2\sigma_X^2}$$

Property 1

Linear combinations of J Gaussian r.v.'s are jointly Gaussian's.

X, Y are jointly gaussian.

$$\phi_{X,Y}(w_1,w_2) = e^{j(w_1\bar{X}+w_2\bar{Y}) - \frac{1}{2}(w_1^2\sigma_X^2 + w_2^2\sigma_Y^2 + 2w_1w_2\sigma_X\sigma_Y\rho_{XY})}$$

$$\begin{cases} Z = aX+bY \\ W = cX+dY \end{cases}$$

$E[Z] = \mu_Z = a\mu_X + b\mu_Y$

$\mu_X = \bar{X}$, $\mu_Y = \bar{Y}$

$Var[Z] = \sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,Cov[X,Y]$

$= a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_X\sigma_Y\rho_{XY}$

$$\phi_Z(w) = E[e^{jwZ}]$$

$$= E[e^{jw(aX+bY)}]$$

$$= E[e^{j[X(aw)+Y(bw)]}]$$

$$= \phi_{X,Y}(aw, bw)$$

$$= e^{j(aw\bar{X}+bw\bar{Y}) - \frac{1}{2}(a^2w^2\sigma_X^2 + b^2w^2\sigma_Y^2 + 2abw^2\sigma_X\sigma_Y\rho_{XY})}$$

$$= e^{j\underbrace{(a\mu_X+b\mu_Y)}_{\mu_Z}w - \frac{1}{2}\underbrace{(a^2\sigma_X^2 + 2\rho ab\sigma_X\sigma_Y + b^2\sigma_Y^2)}_{\sigma_Z^2}w^2}$$

$$= e^{j\mu_Z w - \frac{1}{2}\sigma_Z^2 w^2}$$

(CHF of a typical Gaussian R.V.)

$Z \sim N(a\mu_X + b\mu_Y,\ \sigma_Z^2)$

Similarly $W \sim N(c\mu_X + d\mu_Y,\ \sigma_W^2)$

(property 2 proved)

$$\Phi_{Z,W}(\omega_1,\omega_2) = E\left[e^{j(Z\omega_1+W\omega_2)}\right]$$

$$= E\left[e^{j[(aX+bY)\omega_1+(cX+dY)\omega_2]}\right]$$

$$= E\left[e^{j[(a\omega_1+c\omega_2)X+(b\omega_1+d\omega_2)Y]}\right]$$

$$= \Phi_{X,Y}(a\omega_1+c\omega_2,\ b\omega_1+d\omega_2)$$

$$= \exp\left\{j\left[\underbrace{(a\omega_1+c\omega_2)}_{u}\bar{X}+\underbrace{(b\omega_1+d\omega_2)}_{v}\bar{Y}\right] - \frac{1}{2}\left(\sigma_X^2u^2+\sigma_Y^2v^2+2uv\sigma_X\sigma_Y\rho_{XY}\right)\right\}$$

$$= \exp\left\{j\left[\underbrace{(a\bar{X}+b\bar{Y})}_{\mu_Z}\omega_1+\underbrace{(c\bar{X}+d\bar{Y})}_{\mu_W}\omega_2\right] - \frac{1}{2}\left(\sigma_Z^2\omega_1^2+\sigma_W^2\omega_2^2+2\omega_1\omega_2\sigma_Z\sigma_W\rho_{ZW}\right)\right\}$$

$$\Downarrow$$

jointly Gaussian.

Property 4

independent $\Rightarrow$ uncorrelated. (✓)

uncorrelated $\overset{?}{\Rightarrow}$ independent

For jointly Gaussian R.V.'s X, Y

$$f_{X,Y}(x,y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\cancel{\rho_{XY}^2}}}$$

$$\exp\left[-\frac{1}{2(1-\cancel{\rho_{XY}^2})}\left(\frac{(x-\bar{X})^2}{\sigma_X^2} - \cancel{\frac{2\rho_{XY}(x-\bar{X})(y-\bar{Y})}{\sigma_X\sigma_Y}} + \frac{(y-\bar{Y})^2}{\sigma_Y^2}\right)\right]$$

uncorrelated $\Rightarrow$ $Cov[X,Y]=0$

$Cov[X,Y] = \rho_{XY} \cdot \sigma_X \cdot \sigma_Y$

$\Rightarrow \rho_{XY} = 0$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y} \exp\left\{-\frac{1}{2}\left(\frac{(x-\bar{X})^2}{\sigma_X^2} + \frac{(y-\bar{Y})^2}{\sigma_Y^2}\right)\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left\{-\frac{(x-\bar{X})^2}{2\sigma_X^2}\right\}$$

$$\cdot \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left\{-\frac{(y-\bar{Y})^2}{2\sigma_Y^2}\right\}$$

$$= a(x) \cdot b(Y)$$

$\Rightarrow$ independent.

$$ax+b = \bar{Y} + \frac{\sigma_Y}{\sigma_X} \rho_{XY} (x - \bar{X})$$

$$\Downarrow$$

$$\begin{cases} a = \frac{\sigma_Y}{\sigma_X} \\ b = \bar{Y} - \frac{\sigma_Y}{\sigma_X} \bar{X} \end{cases}$$