commutative $\begin{cases} A \cup B = B \cup A \\ A \cap B = B \cap A \end{cases}$

associative $\begin{cases} A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C \\ A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C \end{cases}$

distributive $\begin{cases} A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \end{cases}$

Demorgan's laws $\begin{cases} \overline{A \cup B} = \bar{A} \cap \bar{B} \\ \overline{A \cap B} = \bar{A} \cup \bar{B} \end{cases}$

☆ $A - B = A \cap \bar{B}$

mutually exclusive $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$ where $Pr(\emptyset) = 0$

property : $Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} Pr(A_i)$

collectively exhaustive : $\bigcup_\alpha A_\alpha = S$ where $Pr(S) = 1$

$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

Conditional Probability. $Pr(A | B) = \frac{Pr(A \cap B)}{Pr(B)}$ (if $Pr(B) > 0$)

Bayes Rule $Pr(A | B) = \frac{Pr(A \cap B)}{Pr(B)} = \frac{Pr(B|A)\, Pr(A)}{Pr(B)}$

Total Probability Theorem $Pr(B) = \sum_i Pr(B | A_i) Pr(A_i)$

if $\{A_i\}_{i=1}^{\infty}$ is mutually exclusive and collectively exhaustive.

Indepence. A, B are indep. $\Leftrightarrow Pr(A \cap B) = Pr(A) \cap Pr(B)$

$\Leftrightarrow Pr(A|B) = Pr(A)$

$\Leftrightarrow Pr(B|A) = Pr(B)$

$n \geq 2$ events. $\{A_i\}_{i=1}^{n}$ are indep. $\Leftrightarrow Pr(A_{i_1} \cap \cdots \cap A_{i_k}) = Pr(A_{i_1}) \cdots Pr(A_{i_k})$

for all $1 \leq i_1 < \cdots < i_k \leq n$ and all $2 \leq k \leq n$

$n=3$ $\quad A, B, C$ indep. $\iff \begin{cases} Pr(A\cap B\cap C) = Pr(A)Pr(B)Pr(C) \\ \begin{cases} Pr(A\cap B) = Pr(A)Pr(B) \\ Pr(B\cap C) = Pr(B)Pr(C) \\ Pr(A\cap C) = Pr(A)Pr(C) \end{cases} \end{cases}$

Bernoulli Trials

$P_n(k) := Pr(A \text{ occurs on } k \text{ of } n \text{ trials}) = \binom{n}{k} p^k (1-p)^{n-k}$

$k = 0, \cdots, n \qquad p = Pr(A)$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

cdf $\quad F_X(x) = Pr(X \le x) = \int_{-\infty}^{x} f_X(x')\,dx'$

pdf $\quad f_X(x) = \frac{d}{dx} F_X(x)$

$\qquad Pr(X \in A) = \int_A f_X(x)\,dx$

pmf $\quad P_X(x) = Pr(X = x)$

Function of R.V.

① X is discrete. $\quad Y = g(X) \quad$ pmf of X $P_X(x)$

$$P_Y(y) = \sum_{x: g(x)=y} P_X(x) = \sum_{x: g(x)=y} P_X(g^{-1}(y))$$

\# of summation term = \# of mapped x_i for a y_i

$\qquad = n$ if $g(x)$ is n-to-1 func.

② X is cont.

Density Method. $\quad f_Y(y) = f_1(y) + f_2(y)$

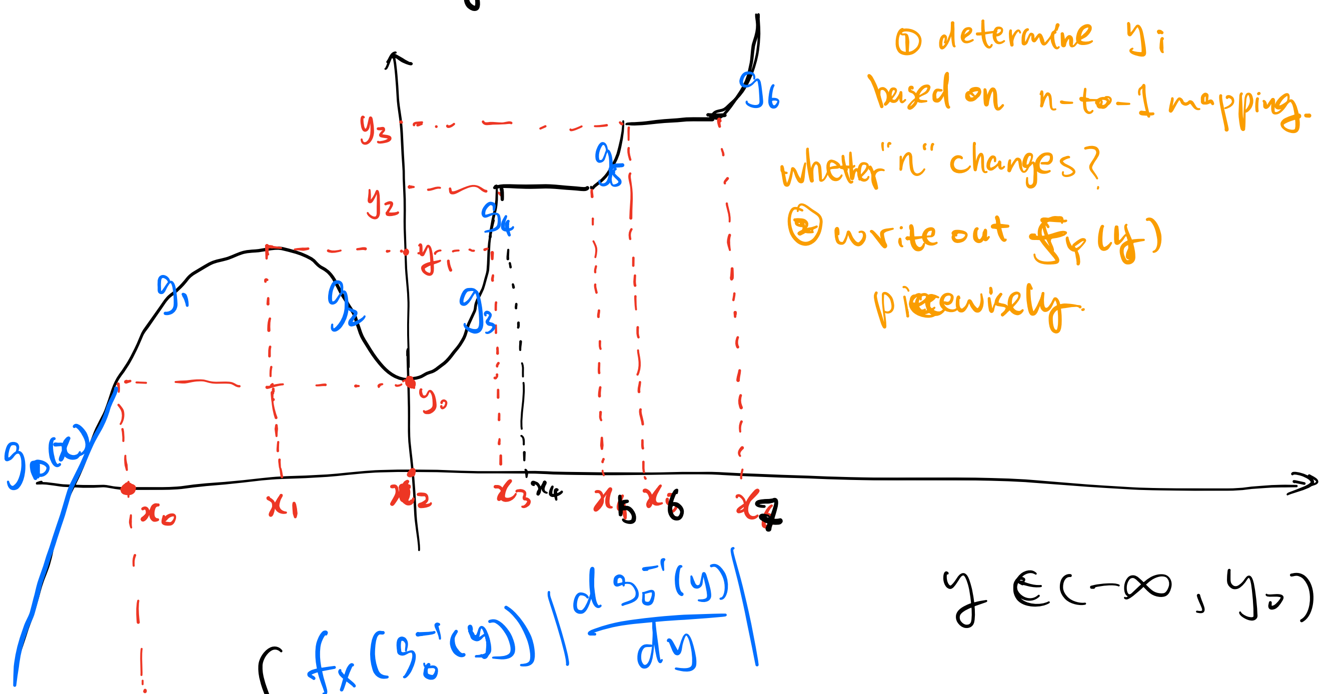
$$f_1(y) = \begin{cases} \sum_i f_X(x_i) \left|\frac{dx_i}{dy}\right|, & y = g(x_1) = g(x_2) = \cdots, \\ 0, & \text{else} \end{cases}$$

$$f_2(y) = \sum_{j=1}^{n} Pr(Y = y_j) \cdot \delta(y - y_j)$$

$$= \sum_{j=1}^{n} \Pr\Big(X \in \bigcup_k (a_{jk}, b_{jk})\Big) \cdot \delta(y - y_j)$$

$$\overset{\{(a_{jk},\, b_{jk})\}}{\underset{\text{disjoint}}{=\!=}} \sum_{j=1}^{n} \sum_{k=1}^{m} \Pr\big(X \in (a_{jk}, b_{jk})\big) \cdot \delta(y - y_j)$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{m} \int_{a_{jk}}^{b_{jk}} f_X(x)\,dx \cdot \delta(y - y_j)$$



① determine y_i based on n-to-1 mapping. whether "n" changes?

② write out $f_Y(y)$ piecewisely

$$f_Y(y) = \begin{cases} f_X(g_0^{-1}(y)) \left| \dfrac{d g_0^{-1}(y)}{dy} \right| & y \in (-\infty, y_0) \\[2ex] \displaystyle\sum_{i=1}^{3} f_X(g_i^{-1}(y)) \left| \dfrac{d g_i^{-1}(y)}{dy} \right| & , y \in (y_0, y_1) \\[2ex] f_X(g_4^{-1}(y)) \left| \dfrac{d g_4^{-1}(y)}{dy} \right| & , y \in (y_1, y_2) \\[2ex] \Pr(Y = y_2) \cdot \delta(y - y_2) \\ \quad = \Pr(x_4 < X < x_5) \cdot \delta(y - y_2) \\ \quad = \displaystyle\int_{x_4}^{x_5} f_X(x)\,dx \cdot \delta(y - y_2) & , y = y_2 \\[2ex] f_X(g_5^{-1}(y)) \left| \dfrac{d g_5^{-1}(y)}{dy} \right| & , y \in (y_2, y_3) \\[2ex] \Pr(Y = y_3) \cdot \delta(y - y_3) & , y = y_3 \end{cases}$$

$$\begin{cases} f_X(g_6^{-1}(y)) \left| \frac{d g_6^{-1}(y)}{dy} \right| , & y \in (y_3, \infty) \\ 0 , & \text{else} \end{cases}$$

Distribution Method

$$F_Y(y) = Pr(Y \le y) = Pr(g(X) \le y) = \int_{g(x) \le y} f_X(x)\,dx$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

solve value range of x

$x \in [h_1(y), h_2(y)]$

$$E[X] = \begin{cases} \int_{-\infty}^{\infty} x f_X(x)\,dx \\ \sum_x x\, p_X(x) \end{cases}$$

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \\ \sum_x g(x)\, p_X(x) \end{cases}$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$E[aX + b] = aE[X] + b$$

$$E[g(X) + h(X)] = E[g(X)] + E[h(X)]$$

$$Var[aX + b] = a^2 Var[X]$$

Condition pdf

$$f_X(x \mid X \in A) = \frac{f_X(x) I_A(x)}{Pr(X \in A)}$$

$$p_X(x \mid X \in A) = \frac{p_X(x) I_A(x)}{Pr(X \in A)}$$

Total

cdf $F_X(x) = \sum_j F_X(x|M_j) \Pr(M_j)$

pdf $f_X(x) = \sum_j f_X(x|M_j) \Pr(M_j)$

pmf $p_X(x) = \sum_j p_X(x|M_j) \Pr(M_j)$

expectation $E[X] = \sum_j E[X|M_j] \Pr(M_j)$

Variance $\mathrm{Var}[X] = \sum_j \left(\mathrm{Var}[X|M_j] + (E[X|M_j] - E[X])^2\right) \cdot \Pr(M_j)$