1. $Pr(A-B) = Pr(A \cap \bar{B})$

$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

$\Rightarrow Pr(A \cup \bar{B}) = Pr(A) + Pr(\bar{B}) - Pr(A \cap \bar{B})$

$\Rightarrow Pr(A \cap \bar{B}) = Pr(A) + Pr(\bar{B}) - Pr(A \cup \bar{B})$

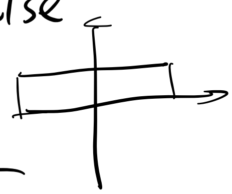
2. $\sum_{i=1}^{n} Pr(B|A_i) \cdot Pr(A_i) = \sum_{i=1}^{n} Pr(B \cap A_i)$

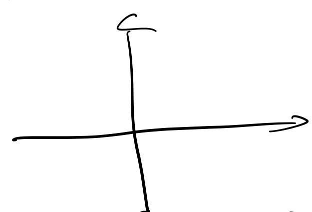
mutual $\quad Pr(B \cap (\bigcup_{i=1}^{n} A_i)) \le Pr(B \cap S) = Pr(B)$

$\bigcup_{i=1}^{n} A_i \subseteq S$, where $Pr(S) = 1$

3. $f_X(x | X \in A) = \dfrac{f_X(x) \cdot I_A(x)}{Pr(x \in A)} = \begin{cases} \dfrac{f_X(x)}{Pr(x \in A)}, & x \in A \\ 0, & \text{else} \end{cases}$

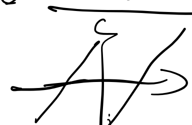
$I_A(x) = \begin{cases} 1, & x \in A \\ 0, & \text{else} \end{cases}$





4. $X \sim U(-5,5)$

$q(x) = \begin{cases} -2.5 & -5 \le x < 0 \\ 2.5 & 0 \le x \le 5 \end{cases}$ $\quad g(x) = X - q(X) = \begin{cases} x+2.5, & x \in [-5,0) \\ x-2.5, & x \in [0,5] \end{cases}$



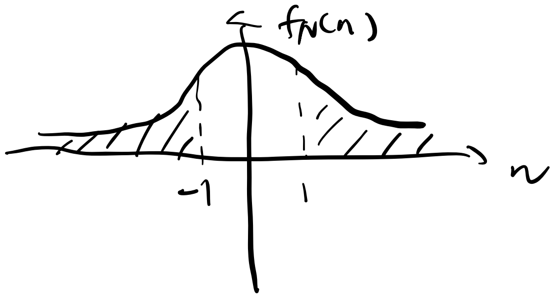
$Var[X - q(X)] = \frac{1}{12}(b-a)^2 = \frac{1}{12}[2.5-(-2.5)]^2 = \frac{25}{12}$

$X - q(X) \sim U[-2.5, 2.5]$

5. $Pr(e) = Pr(\text{absent} \mid \text{present}) \cdot Pr(\text{present})$
$\quad + Pr(\text{present} \mid \text{absent}) \cdot Pr(\text{absent})$

$= Pr(2+N \le 1) \cdot \frac{1}{2} + Pr(N > 1) \cdot \frac{1}{2}$

$= Pr(N \le -1)\cdot\frac{1}{2} + Pr(N>1)\cdot\frac{1}{2}$



$= 2\cdot Pr(N \le -1)\cdot\frac{1}{2} = \Phi(-1)$

$= 2\cdot Pr(N>1)\cdot\frac{1}{2} = 1-\Phi(1)$

6. $\hat{x}_{MAP} = \arg\max_{x} f_{X|Y}(x|y)$

$X \sim N(1,1)$, $N \sim N(0,1)$

$\begin{cases} Y = X+N \\ X = X \end{cases} \Rightarrow X, Y$ joint Gaussian

$f_{X|Y}(x|y)$ is a conditional Gaussian pdf.

$\hat{x}_{MAP} = m_{X|Y}(y) = \bar{X} + \frac{\sigma_X}{\sigma_Y}\rho_{XY}(y-\bar{Y})$

$= 1 + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(y-1) = \frac{1}{2}y + \frac{1}{2}$

$Var[Y] = Var[X+N] \overset{indep}{=} Var[X] + Var[N] = 2$

$E[Y] = \bar{Y} = E[X+N] = E[X] + E[N] = 1$

$\rho_{XY} = \frac{Cov[X,Y]}{\sigma_X\sigma_Y} = \frac{\underbrace{Cov[X, X+N]}_{= Cov[X,X] + Cov[X,N]}}{\sigma_X\sigma_Y} = \frac{\underline{Var[X] + Cov[X,N]}}{\sigma_X\sigma_Y}$

$$= \frac{Var[X]}{\sigma_X \sigma_Y} = \frac{1}{\sigma_Y} = \frac{1}{\sqrt{2}}$$

7. Memoryless

8. $Cov[U,V] = Cov[X+Y,\ X-Y]$

$$= Cov[X,X] - Cov[X,Y] + Cov[Y,X] - Cov[Y,Y]$$

$$= Var[X] - Var[Y]$$

9. $$\begin{cases} U = X+Y \\ V = X-Y \end{cases} \Rightarrow \begin{cases} X = \frac{1}{2}(U+V) \\ Y = \frac{1}{2}(U-V) \end{cases}$$

$$J(x,y) = \left\| \begin{matrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{matrix} \right\| = \left\| \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} \right\| = 2$$

$$f_{U,V}(u,v) = \frac{1}{J(x,y)} \cdot f_{X,Y}\left(\frac{1}{2}(u+v),\ \frac{1}{2}(u-v)\right)$$

10. $$\begin{cases} \text{①}\ E[X(t)] = const. \\ \text{②}\ E[X(t)X(t+\tau)] = f(\tau) \end{cases}$$

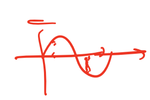
① $E[X(t)] = E[A\cos(\omega t + \Theta)]$

$\Theta \sim U[0,\pi)$, indep of A.

$$f_\Theta(\theta) = \begin{cases} \frac{1}{\pi}, & \theta \in [0,\pi] \\ 0, & \text{else} \end{cases}$$

$$= E[A] \cdot E[\cos(\omega t + \Theta)]$$

$$= \bar{A} \cdot \underbrace{\int_0^\pi \cos(\omega t + \theta) \cdot \frac{1}{\pi}\, d\theta}_{f(t)}$$



$\sin(\theta + \omega t)\Big|_0^\pi$

$\sin(\pi + \omega t) - \sin\omega t \neq const.$

11. $E[X(t)] = E[\sum_{k=-\infty}^{\infty} Y_k \cdot p(t-k\cdot\Delta)]$

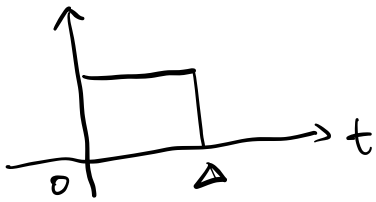
$= \sum_{k=-\infty}^{\infty} E[Y_k \cdot p(t-k\cdot\Delta)]$

$\overset{\text{indep}}{=} \sum_{k=-\infty}^{\infty} E[Y_k] \cdot E[p(t-k\cdot\Delta)]$

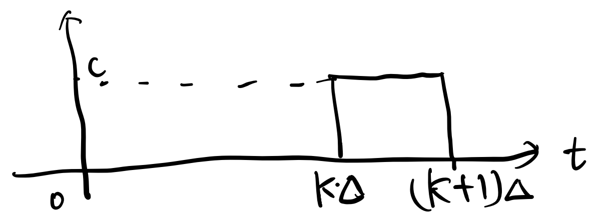
Y_k binary sequence, can be 0 or 1

$E[Y_k]$ can be 0 or 1.

$p(t)$



$p(t-k\cdot\Delta)$



$E[p(t-k\cdot\Delta)] = \begin{cases} c, & t \in [k\cdot\Delta, (k+1)\Delta) \\ 0, & \text{else} \end{cases}$

$\sum_{k: Y_k=1} E[p(t-k\cdot\Delta)] \neq \text{const.}$

$= \begin{cases} c, & t \in [k\cdot\Delta, (k+1)\Delta), \quad k: Y_k=1 \\ 0, & t \notin [k\cdot\Delta, (k+1)\Delta), \quad k: Y_k=1 \end{cases}$

12. $E[X(t)] = \text{const}_1 = c_1$

$Var[X(t)] = E[X^2(t)] - c_1^2 = R_X(0) - c_1^2 = \text{const}_2.$

$E[X^2(t)] = E[X(t)\cdot X(t+0)] = \underline{R_X(0)}$

$R_X(\tau) = E[X(t)\cdot X(t+\tau)]$

13. Lecture 35. Page 202

14. Lecture 35. page 203

15. $X(t) \longrightarrow \boxed{LTI} \longrightarrow Y(t)$

LTI property: Lec 37. page 223

if $X(t)$ is WSS $\Rightarrow \begin{cases} Y(t) \text{ is WSS. r.p.} \\ X(t), Y(t) \text{ are J WSS. r.p.} \end{cases}$

$E[Y^2(t)] = R_Y(0) = \int_{-\infty}^{\infty} e^{i2\pi f\tau} S_Y(f) df$

where $\tau = 0$

$R_Y(\tau) = \mathcal{F}^{-1}(S_Y(f))$

$\parallel$

$\int_{-\infty}^{\infty} S_Y(f) df$

$\parallel$

$S_Y(f) = \underbrace{|H(f)|^2 \cdot S_X(f)}_{\text{proved in Lec 37}}$

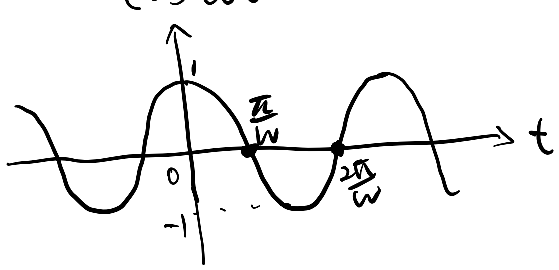
$\int_{-\infty}^{\infty} |H(f)|^2 \cdot S_X(f) df$

17. $\Theta \sim U[0, 2\pi)$, $B \sim U[-a, a]$

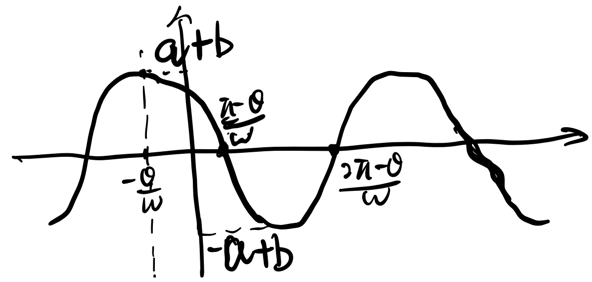
B is indep. of Θ

(a) $X(t) = a\cos(\omega t + \theta) + b$ $\quad t \in (-\infty, \infty)$

$\cos \omega t$



$a\cos(\omega t + \theta)^{+b} = a\cos(\omega(t + \frac{\theta}{\omega}))^{+b}$



(b) $E[X(t)] = E[a\cos(wt+\Theta) + B]$

$= aE[\cos(wt+\Theta)] + E[B]$ $\quad E[B] = 0$

$= a\int_0^{2\pi} \cos(wt+\theta)\cdot\frac{1}{2\pi}d\theta + 0$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & \theta\in[0,2\pi) \\ 0, & \text{else} \end{cases}$$

$\downarrow$

0

$E[X(t)] = 0 + 0 = 0$

$E[X(t)X(t+\tau)] = E[(a\cos(wt+\Theta) + B)(a\cos(w(t+\tau)+\Theta)+B)]$

① ② ③ ④

$= E[a^2\cos(wt+\Theta)\cos(wt+w\tau+\Theta)]$ ⑤

$+ E[B\cdot \cancel{a\cos(wt+\Theta)}] = E[B]\cdot E[a\cos\ldots] = 0$

$+ E[B\cdot \cancel{a\cos(wt+w\tau+\Theta)}] = 0$

$+ E[B^2]$ $\frac{a^2}{3}$

$B \sim U(-a, a)$ $\quad E[B^2] = Var[B] + E^2[B]$

$= Var[B]$

$(\cos x)\cdot(\cos y) = \frac{1}{2}(\cos(x+y)+\cos(x-y))$ $\quad = \frac{1}{12}(a-(-a))^2$

$= \frac{a^2}{3}$

⑤ $= E\left[\frac{1}{2}a^2[\cos(w\tau) + \cos(2wt+2\Theta+w\tau)]\right]$

$\Downarrow$

$\Theta \sim U[0, 2\pi)$ $\quad E[\cos(2wt+2\Theta+w\tau)] = 0$

$E[\cos(w\tau)] = \int_0^{2\pi} \cos(w\tau)\frac{1}{2\pi}d\theta$ $\rightarrow$ pdf of Θ

$$\frac{1}{2\pi}\cos(\omega\tau)\theta\Big|_0^{2\pi} = \cos(\omega\tau).$$

$$E[X(t)X(t+\tau)] = \frac{1}{2}a^2\cos\omega\tau + \frac{a^2}{3} = f(\tau)$$

$$\begin{cases} m_X = E[X(t)] = 0 \\ R_X(\tau) = E[X(t)X(t+\tau)] = \frac{1}{2}a^2\cos\omega\tau + \frac{a^2}{3} \end{cases}$$

(c) $E[X(t_1)X(t_2)] = E[X(t_1)]E[X(t_2)]$

$$R_X(t_1-t_2) = m_X \cdot m_X$$

$$\underline{\frac{1}{2}a^2\cos\omega(t_1-t_2) + \frac{a^2}{3} = 0}$$

$$\Downarrow$$

$$\cos\omega(t_1-t_2) = -\frac{2}{3}$$

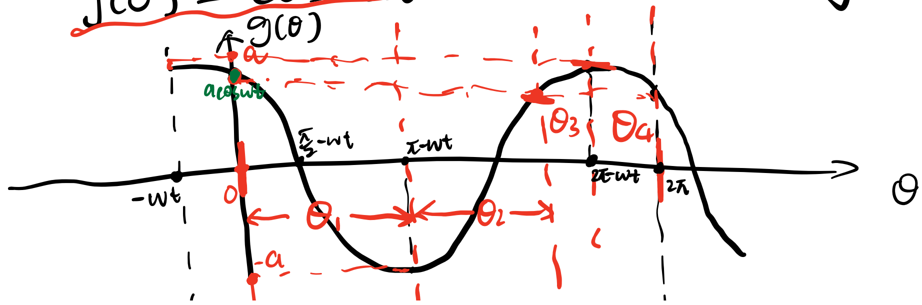
$$\downarrow$$

$$[-1,1] \Rightarrow \text{solvable (have solution)}$$

(d) <u>find $f_{Y(t)}(y)$</u> $\quad f_{\Theta}(\theta) = \begin{cases} \frac{1}{2\pi}, & \theta \in (0, 2\pi) \\ 0, & \text{else.} \end{cases}$

$$Y(t) = a\cos(\omega t + \Theta) = g(\Theta)$$

$g(\theta) = a\cos(\omega t + \theta)$ $\qquad$ Density method



$$f_{Y(t)}(y) = \begin{cases} \sum_{i=1,2} f_{\Theta}(\theta_i) \cdot \frac{1}{|g'(\theta_i)|} & y \in (-a, a\cos\omega t) \\ \sum_{i=3,4} f_{\Theta}(\theta_i) \cdot \frac{1}{|g'(\theta_i)|}, & y \in (a\cos\omega t, a) \end{cases}$$

$$a\sin(\omega t+\theta) = \sqrt{a^2-y^2}$$

$$g'(\theta) = -a\sin(\omega t+\theta) = -\sqrt{a^2-y^2}$$

$$f_{Y(t)}(y) = \begin{cases} 2\cdot\frac{1}{2\pi}\cdot\frac{1}{\sqrt{a^2-y^2}}, & y \in (-a, a\cos\omega t) \\ 2\cdot\frac{1}{2\pi}\cdot\frac{1}{\sqrt{a^2-y^2}}, & y \in (a\cos\omega t, a) \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \frac{1}{\pi}\frac{1}{\sqrt{a^2-y^2}}, & y \in (-a, a) \\ 0, & \text{else} \end{cases}$$

$$X(t) = Y(t) + B$$

$$\begin{cases} B, \Theta & \text{indep} \\ Y(t) = g(\Theta) \end{cases}$$

B, $g(\Theta)$ indep

$\Downarrow$

$Y(t)$, B indep

$$f_{X(t)}(x) = f_B * f_{Y(t)}(x) = \underline{\int_{-\infty}^{\infty} f_B(b)\cdot f_{Y(t)}(x-b)\,db}$$

$$\parallel$$

$$f_B(b)=\begin{cases}\frac{1}{2a}, & b\in(-a,a)\\ 0, & \text{else}\end{cases}$$

$$\Downarrow$$

$$\int_{-a}^{a}\frac{1}{2a}\cdot f_{Y(t)}(x-b)\,db$$

$$\int_{-a}^{a}\frac{1}{\pi\sqrt{a^2-(x-b)^2}}\cdot\frac{1}{2a}\,db$$

def: $c=b-x$

$$=\int_{-a-x}^{a-x}\frac{1}{2a\pi\sqrt{c^2-a^2}}\,dc$$

$$=\frac{1}{2a\pi}\int_{-a-x}^{a-x}\frac{1}{\sqrt{c^2-a^2}}\,dc$$

$$=\frac{1}{2a\pi}\left(\sin^{-1}\left(\frac{c}{a}\right)\right)\Big|_{-a-x}^{a-x}$$

$$=\frac{1}{2a\pi}\left(\sin^{-1}\left(\frac{a-x}{a}\right)-\sin^{-1}\left(\frac{-a-x}{a}\right)\right)$$

$$=\frac{1}{2a\pi}\left(\sin^{-1}\left(\right.\right.$$

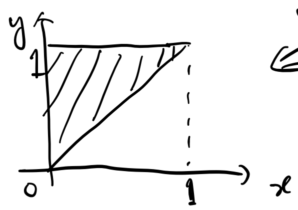
Question 17 (f) find $f_Z(z)$

Solution 1: Distribution Method

Given $\begin{cases} f_{X,Y}(x,y) = \begin{cases} 2, & 0 \le x \le y \le 1 \\ 0, & \text{else} \end{cases} \\ Z = X+Y \end{cases}$

CDF: $F_Z(z) = Pr(Z \le z) = Pr(X+Y \le z) = \iint_{x+y \le z} f_{X,Y}(x,y)\,dx\,dy$

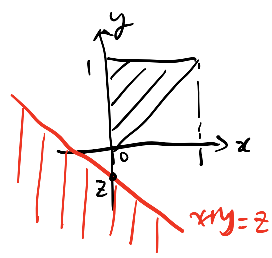
By sketching the possible values of (x,y) from $0 \le x \le y \le 1$ we get a triangle



$x+y \le z$ means: the area below the line $x+y=z$

So there are 4 cases

case 1 $z \le 0$, $\Rightarrow$

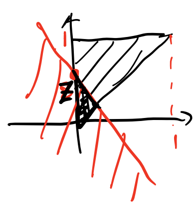


There is no intersection between the red shade and black shade

Which means: $\iint_{x+y \le z} f_{X,Y}(x,y)\,dx\,dy = 0$

In other words, $Pr(X+Y \le z) = 0$

Case 2 $0 < z \le 1$



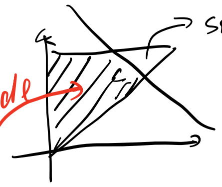
The intersection area is a small triangle.

$Pr(X+Y \le z) = \iint_{x+y \le z} f_{X,Y}(x,y)\,dx\,dy = \iint_{x+y \le z} 2\,dx\,dy = 2 \iint_{x+y \le z} dx\,dy$

means the area of the small triangle

$= 2 \cdot \frac{1}{2} \cdot \left(\frac{z}{\sqrt{2}}\right)^2$

Case 3 $\quad 1 < Z \le 2$



$Pr(X+Y \le z) = 2 \iint_{x+y\le z} dx\,dy$ area of shade

$= 2 \cdot (\text{big triangle} - \text{small triangle})$

$= 2 \cdot \left(\frac{1}{2} \cdot 1 \cdot 1 - \frac{1}{2} \cdot \left(\frac{2-z}{\sqrt{2}} \right)^2 \right)$

Case 4 $\quad Z > 2$

$Pr(X+Y \le z) = 2 \cdot \frac{1}{2} = 1.$

Solution 2. Density Method.

$\begin{cases} Z = X+Y \\ W = Y \end{cases}$ ⇒ auxiliary variable. ⇒ $\begin{cases} X = Z-W \\ Y = W \end{cases}$

$J(x,y) = \left\| \begin{matrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{matrix} \right\| = \left\| \begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} \right\| = 1$

$f_{Z,W}(z,w) = f_{X,Y}(x(z,w), y(z,w)) \frac{1}{J(x,y)} = f_{X,Y}(z-w, w)$

We know than only when $0 \le x \le y \le 1$, $f_{X,Y}(x,y) = 2$

otherwise, $f_{X,Y}(x,y) = 0$

$f_{Z,W}(z,w)$

$= f_{X,Y}(z-w,w) = \begin{cases} 2, & \text{when } 0 \le z-w \le w \le 1 \\ 0, & \text{else} \end{cases}$

Solve this!

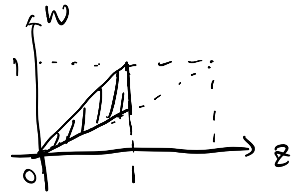
$\begin{cases} 0 \le z-w \le w \\ 0 \le w \le 1 \\ 0 \le z-w \le 1 \\ z-w \le w \le 1 \end{cases} \xrightarrow{\text{solve } w} \begin{cases} \frac{z}{2} \le w \le z & (1) \\ 0 \le w \le 1 & (2) \\ z-1 \le w \le z & (3) \\ \frac{z}{2} \le w \le 1 & (4) \end{cases}$

$f_Z(z) = \int_{\cdots}^{\cdots} f_{Z,W}(z,w)\,dw$

① case 1: $z \le 0$, (1)(2)(3)(4) have no intersection $\Rightarrow w \in \phi$

$\Rightarrow f_Z(z) = \int_{w\in\phi} f_{Z,W}(z,w)\,dw = 0$

② case 2. $0 < z \le 1$, (1)(2)(3)(4) $\Rightarrow \begin{cases} \frac{z}{2} \le w \le z \\ z \in (0,1] \end{cases}$



$f_Z(z) = \int_{\frac{z}{2}}^{z} f_{Z,W}(z,w)\,dw = \int_{\frac{z}{2}}^{z} 2\,dw = 2\cdot\left(z - \frac{z}{2}\right) = z$

③ case 3 $1 < z \le 2$, (1)(2)(3)(4) $\Rightarrow \begin{cases} \frac{z}{2} \le w \le 1 \\ z \in [1,2] \end{cases}$

$\Downarrow$

$z-1 \le \frac{z}{2}$

$f_Z(z) = \int_{\frac{z}{2}}^{1} f_{Z,W}(z,w)\,dw = \int_{\frac{z}{2}}^{1} 2\,dw = 2\cdot\left(1 - \frac{z}{2}\right) = 2-z$

④ Case 4 $z > 2 \Rightarrow \begin{cases} z-1 > 1 \\ \frac{z}{2} > 1 \end{cases} \Rightarrow$ (1)(2)(3)(4) no intersection

$\Downarrow$

$w \in \phi$

$\Rightarrow f_Z(z) = \int_{w\in\phi} f_{Z,W}(z,w)\,dw$

$= 0$